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LIFETIME ASSESSMENT FOR AN IDEAL ELASTOPLASTIC THICK-WALLED SPHERICAL MEMBER UNDER GENERAL MECHANOCHEMICAL CORROSION CONDITIONS

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Abstract. The problem of the equal-rate mechanochemical wear of an elastic-perfectly plastic thick-walled spherical shell under internal and external pressure is solved analytically. The proposed solution allows to assess the time of the initial yielding at the bore of the shell and the time of fully plastic yielding. The obtained formulas are to be used for design purposes and as a benchmark solution for numerical analysis.

1 INTRODUCTION

Spherical members are often used in metal structures including the automobile bodies, aircraft fuselages, ship hulls, pipes etc. Most of these structures are exploited by being subjected to both mechanical loads and operating environments. This often causes the process of so-called mechanochemical wear, which is more intensive than the simple superposition of damages induced by mechanical stresses and electrochemical corrosion taken separately. There is a great necessity for an accurate prediction of material loss and rational estimating of the lifetime of structure members under mechanochemical corrosion conditions. Corrosion can be concentrated locally, or it can extend across a wide area more or less uniformly corroding the surface. This paper is concerned with equal-rate mechanochemical corrosion often observed in practice. The theoretical studies in this area have been conducted by numerous authors. Comprehensive reviews of models and calculations for structures taking into account uniform corrosive wear have been done e.g., in [1]. Among the first works in the field are articles by V.M. Dolinskii concerned with mechanochemical corrosion of thin-walled structural members [2]. In [3] the lifetime of loaded spherical shells was assessed under the assumption of the exponential dependence of the rate of corrosion on the average normal stress. Some researchers have simulated the corrosive wear of structure elements using the linear relation between the corrosion rate and the equivalent stress [1, 2, 4, 5, 6, ...]. This dependance is often observed in experiments [1, 2, 7]. All the related coefficients depend on properties of the "material–medium" system. The mechanochemical effect of the strain sign [8] can be taken into consideration by using different observable constants in the relationships for corrosion rates. In the majority of cases, the problems of mechanochemical corrosion have been studied by numerical methods. There are only few works where analytical solutions have been found. Most of them are devoted to thin-walled structure members [2, 4]. In such articles [5, 6, 9] the closed-form solutions for the problems of the equal-rate mechanochemical corrosion of linearly elastic and elastic-plastic thick-walled cylindrical tubes have been obtained. The purpose of this work is to present an analytical solution for the problem of the general mechanochemical corrosion of an ideal elastic-plastic thick-walled spherical shell under pressure using the linear relation between the corrosion rate and the stress intensity. We begin in Section 2 by formulating the problem. In Section 3, we derive the closed-form solutions for the pure elastic stage. In Section 4, we analyze the behavior of corroded vessel during the partially plastic stage. The time of plastic-zone propagation through the shell is determined there. Proposed analytical solution is useful for design purposes and as a benchmark solution for numerical analysis.

2 PROBLEM SPECIFICATION

Consider a concentric thick-walled spherical shell, with inner radius r and outer radius R changing with time t and loaded with internal pressure p_r and external pressure p_R of corrosive environments. The inner and outer radii at the initial time $t = 0$ are denoted by r_0 and R_0, respectively. Since $t = 0$, the solid is subjected to uniform mechanochemical wear. The corrosion rates at the internal and external boundaries are described correspondingly by the expressions [1, 2, 7]:

$$v_r = \frac{dr}{dt} = a_r + m_r \sigma_i(r), \tag{1}$$

$$v_R = -\frac{dR}{dt} = a_R + m_R \sigma_i(R). \tag{2}$$

Here, a_r, a_R, m_r, m_R are observable quantities;

$$\sigma_i(\rho) = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{3}$$

is the stress intensity; σ_1, σ_2, and σ_3 are principal stresses; $r \leq \rho \leq R$.

The sphere's material is modeled as a linear elastic, perfectly plastic material with the yield stress σ_y. The condition of yielding for an ideally plastic substance is expressed by the von Mises-Hencky energy criterion

$$\sigma_i = \sigma_y. \tag{4}$$

It is necessary to trace the stress intensity with time t and to assess the lifetime of the vessel without an allowance for buckling.

3 SOLUTION FOR THE PURE ELASTIC STAGE

Let the shell under pressure be at first elastic, i.e., the stress intensity σ_i throughout the body is below the yield stress for $t \in [0, t^e[$, $(t^e > 0)$. With reference to the spherical coordinates ρ, θ, φ (with the origin at the center of the sphere), the stress-components at this stage are expressed by G. Lame's solution for a pressurized thick-walled spherical shell [10]. From the spherical symmetry, it follows that, at any point the radial and both tangential directions are principal axes of stresses: $\sigma_1 = \sigma_2 = \sigma_{\theta\theta} = \sigma_{\varphi\varphi}$ and $\sigma_3 = \sigma_{\rho\rho}$. Therefore, Eq. (3) can be written in the form

$$\sigma_i(\rho) = |\sigma_{\theta\theta}(\rho) - \sigma_{\rho\rho}(\rho)|. \tag{5}$$

Substituting the Lame solution into Eq. (5), the stress intensity at the inner and outer surfaces at any $t \in [0, t^e[$ can be found as

$$\sigma_i(r) = \frac{3\Delta p}{2} \frac{\eta^3}{\eta^3 - 1}, \tag{6}$$

$$\sigma_i(R) = \frac{3\Delta p}{2} \frac{1}{\eta^3 - 1}, \tag{7}$$

where $\Delta p = |p_r - p_R|$ and

$$\eta = \frac{R}{r}. \tag{8}$$

It is evident that the stress intensity is at maximum at the inner surface:

$$\sigma_i(r) = \max_{r \le \rho \le R} \sigma_i(\rho).$$

Hence, yielding begins at the bore of the sphere, so we must follow the $\sigma_i(r)$ to determine the time t^e of the end of the pure elastic stage. Let $\sigma_i(r)$ be denoted by $\sigma_{i,r}$.

From Eq. (6), using the denotation $\sigma_{i,r} = \sigma_i(r)$, the ratio η is expressed as

$$\eta = \sqrt[3]{\frac{\sigma_{i,r}}{\sigma_{i,r} - 3\Delta p/2}}. \tag{9}$$

Eqs. (6)–(9) are valid until the time t^e at which the metal at the bore of the sphere becomes plastic.

The initial conditions to be satisfied at $t = 0$ are

$$\sigma_{i,r}^0 = \sigma_i(r)\big|_{t=0} = \frac{3\Delta p}{2} \frac{\eta_0^3}{\eta_0^3 - 1}, \qquad \eta_0 = \frac{R_0}{r_0}. \tag{10}$$

3.1 Case of homogeneous stress

If $\{r = 0,\, p_R = p\}$ or $\{r \neq 0,\, p_r = p_R = p\}$, the stress state in the sphere is homogeneous

$$\sigma_{\rho\rho} \equiv \sigma_{\theta\theta} \equiv \sigma_{\varphi\varphi} \equiv -p, \qquad \sigma_i = 0$$

at any time and irrespective of corrosion. In these cases, the stress intensity σ_i equals zero and never reaches the yield point up to complete dissolution. The vessel thickness $h = R - r$ at any time is then

$$h = h_0 - t\,(a_r + a_R),$$

where $h_0 = R_0 - r_0$ is the initial thickness of the shell.

After that, the sphere's lifetime can be determined as the time to complete dissolution:

$$t^d = \frac{h_0}{a_r + a_R}.$$

3.2 Case of nonhomogeneous stress

Now consider rather complicated situations when $p_r \neq p_R$. If the corrosion rates $\dfrac{dr}{dt}$, $\dfrac{dR}{dt}$ depend respectively on the stresses $\sigma_i(r)$ and $\sigma_i(R)$, and one of them, in its turn, depends on both the variable sphere radii r and R, it is impossible to obtain simple explicit functions $r(t)$ and $R(t)$, and thus the expressions for $\sigma_i(\rho)$. Let us eliminate the variables r, R, η, and $\sigma_i(R)$ from the system of equations (1), (2), and (6)–(8) and attempt to derive a relationship between the maximum stress intensity $\sigma_{i,r}$ and the time t.

Eliminating $\sigma_i(r)$ and $\sigma_i(R)$ from Eqs. (1) and (2), by means of the relationships given in Eqs. (6) and (7), yields

$$r = \frac{m_R r_0 + m_r R_0 + (m_R a_r - m_r a_R + 3 m_r m_R\, \Delta p/2)\, t}{\eta\, m_r + m_R}. \tag{11}$$

Differentiating Eq. (6) with respect to t and following transformation by the use of Eqs. (1), (2) and (8), (9), (11) gives the ordinary differential equation for changing $\sigma_{i,r}$ in the form

$$\frac{d\sigma_{i,r}}{dt} = 2\, \frac{\sqrt[3]{(\sigma_{i,r})^2 (\sigma_{i,r} - 3\Delta p/2)^2}}{\Delta p} \left(m_r \sqrt[3]{\sigma_{i,r}} + m_R \sqrt[3]{\sigma_{i,r} - 3\Delta p/2} \right) \times$$
$$\times \frac{[a_R + m_R(\sigma_{i,r} - 3\,\Delta p/2)]\sqrt[3]{\sigma_{i,r} - 3\Delta p/2} + [a_r + m_r \sigma_{i,r}]\sqrt[3]{\sigma_{i,r}}}{m_R r_0 + m_r R_0 + (m_R a_r - m_r a_R + 3 m_r m_R\, \Delta p/2)\, t}. \tag{12}$$

The solution of this equation can be obtained by separating variables and integrating. The integral of Eq. (12), satisfying the condition (10) is

$$t = (m_R r_0 + m_r R_0)\ \frac{\exp\left[(m_R a_r - m_r a_R + 3m_r m_R\,\Delta p/2)\,J(\sigma_{i,r})\right] - 1}{m_R a_r - m_r a_R + 3m_r m_R\,\Delta p/2}, \tag{13}$$

where

$$J(\sigma_{i,r}) = \frac{\Delta p}{2}\int\limits_{\sigma_{i,r}^0}^{\sigma_{i,r}} \frac{1}{\sqrt[3]{\sigma^2(\sigma-3\Delta p/2)^2}\left(m_r\sqrt[3]{\sigma}+m_R\sqrt[3]{\sigma-3\Delta p/2}\right)} \times$$
$$\times \frac{d\sigma}{[a_R+m_R(\sigma-3\Delta p/2)]\sqrt[3]{\sigma-3\Delta p/2}+[a_r+m_r\sigma]\sqrt[3]{\sigma}}.$$

The above integral can be taken analytically, but its expression is rather cumbersome so it is not to be represented here. It can be computed numerically with an arbitrarily small error. For one-sided corrosion, the analytical solution can be simplified.

The solutions of the basic differential equation give us the t-to-$\sigma_{i,r}$ corresponding. Using Eq. (9), we can then calculate the relation η at any time t.

As noted before, plastic flow begins at the inner face of the sphere. Let t^e be the time of the end of elastic stage when $\sigma_{i,r}$ reaches the yield stress σ_y (if it does), i.e.,

$$\sigma_{i,r} = \sigma_y \quad \text{at} \quad t \geq t^e.$$

Let r^e and R^e be the inner and the outer radii corresponding to t^e, so that

$$r(t^e) = r^e, \tag{14}$$

$$R(t^e) = R^e, \tag{15}$$

$$\sigma_i(\rho) < \sigma_y \quad \text{for} \quad r^e < \rho \leq R^e \quad \text{at} \quad t = t^e,$$

and

$$\eta^e = R^e/r^e, \quad h^e = R^e - r^e.$$

In other terms, t^e is defined by the solution (13) with σ_y for $\sigma_{i,r}$. Ratio η^e is calculated by Eq. (9) with σ_y for $\sigma_{i,r}$. Then, r^e is obtained by Eq. (11) for $t = t^e$ and $\eta = \eta^e$, and finally $R^e = \eta^e r^e$.

In the case of brittle fracture the lifetime of a shell can be defined as the time at which the equivalent stress at some point along the vessel reaches the ultimate stress, such as when $\sigma_{i,r} = \sigma_s$, where σ_s is the brittle strength. Time to rupture t^e is then determined by the solutions (13) with σ_s for $\sigma_{i,r}$.

For an elastic-perfectly plastic vessel, the time at which the stress intensity at the inner surface reaches the yield point does not correspond to the breakdown. This is explained by the fact that the plastic material is enclosed within the elastic material, which prevents unrestricted flow. Therefore, durability is to be determined with taking into consideration the period of the plastic-zone propagation throughout the sphere wall.

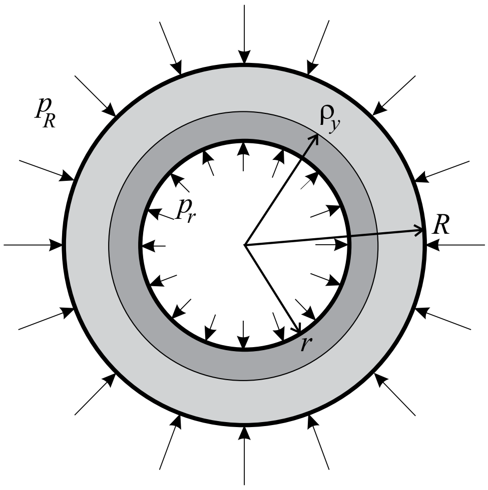

Figure 1: Partially plastic spherical shell

4 SOLUTION FOR THE PARTIALLY PLASTIC STAGE — LIFETIME ASSESSMENT

As the vessel thickness decreases below h^e, a zone of yielding moves outward from the bore through the sphere. According to [11, 12], there is a spherical bounding surface which divides the inner and plastic material from the outer and still elastic material. In the material already yielded the stress-components change their values continuously. For strains in both the elastic and partially plastic sphere, the deformations are assumed to be negligible with respect to the dimensions of a deforming element. Let $\rho_y = \rho_y(t), \quad (t \geq t^e)$ denote the instantaneous value of the radius of the plastic front as shown in Fig. 1. Let also the pressure of the elastic part on the plastic part be denoted by $q = -\sigma_{\rho\rho}(\rho_y)$.

4.1 Fully plastic yielding condition

To investigate the progress of the plastic-elastic boundary from the increasing inner face toward the decreasing outer face of the vessel in the process of corrosion, consider the state of stress for instantaneous values of r, R, $\rho_y : r < \rho_y < R$, as it was performed by [11, 12], and others.

Conditions throughout the inner plastic region ($r \leq \rho \leq \rho_y$) are governed by the von Mises-Hencky yield condition, expressed by Eq. (4). This equation can, by means of Eqs. (5), be reduced to

$$|\sigma_{\theta\theta} - \sigma_{\rho\rho}| = \sigma_y.$$

Substituting the above relationship into the equation of equilibrium of an element of the sphere

$$\frac{d\sigma_{\rho\rho}}{d\rho} + 2\,\frac{\sigma_{\rho\rho} - \sigma_{\theta\theta}}{\rho} = 0,$$

one can obtain differential equation

$$\rho \left| \frac{d\sigma_{\rho\rho}}{d\rho} \right| = 2\,\sigma_y.$$

The solution of this equation is

$$|\sigma_{\rho\rho}| = 2\,\sigma_y \ln \rho + C, \tag{16}$$

where C is the integration constant. The boundary conditions for the plastic-zone are

$$\sigma_{\rho\rho}(r) = -p_r, \quad \sigma_{\rho\rho}(\rho_y) = -q.$$

Entering with these values into Eq. (16), one finds that the following equation holds at any t

$$|q - p_r| = 2\,\sigma_y \ln \frac{\rho_y}{r}. \tag{17}$$

For the outer elastic shell ($\rho_y \leq \rho \leq R$), the expressions for the stresses are obtained from Eqs. (6)–(7) by putting $r = \rho_y$, $p_r = q$, and $\Delta p = |q - p_R|$. In particular, the expression for the stress intensity (6) can, recalling that $\sigma_i(\rho_y) = \sigma_y$, be rewritten in the form

$$|q - p_R| = \frac{2\sigma_y}{3}\,\frac{(R/\rho_y)^3 - 1}{(R/\rho_y)^3}. \tag{18}$$

The addition of Eqs. (17) and (18) gives

$$|p_r - p_R| = \Delta p = \frac{2\sigma_y}{3}\left(1 - \frac{\rho_y^3}{R^3} + 3\ln\frac{\rho_y}{r}\right). \tag{19}$$

The plastic front continues outward until it coincides with the moving outer surface: $\rho_y = R$. Using R for ρ_y, Eq. (19) becomes

$$\Delta p = 2\,\sigma_y \ln \eta. \tag{20}$$

Therefore, full plasticity is reached when the ratio η equals

$$\eta = \eta^f = \exp\left(\frac{\Delta p}{2\sigma_y}\right). \tag{21}$$

Several solutions have been published for the partially plastic thick-walled sphere under internal pressure. Eq. (19) and (20) for $p_R = 0$ agree fairly well with the formulas presented in [11, 12].

4.2 Lifetime assessment

Since the moment $t = t^e$, $\sigma_i(r) = \sigma_y$ and the corrosion rate v_r is not accelerated by stress increase,

$$v_r = v_y = a_r + m_r \sigma_y. \tag{22}$$

The integral of this equation, satisfying the condition (14) is

$$r = r^e + v_y \,(t - t^e). \tag{23}$$

These equations hold true for $t \geq t^e$.

Consider now the elastic-zone of the sphere $\rho_y \leq \rho \leq R$. As it was mentioned in Section 4.1, the expressions for the stress intensity are obtained from Eqs. (6), (7) by putting $r = \rho_y$ and $\Delta p = |q - p_R|$. Comparing Eqs. (6), (7) for $r = \rho_y$, with taking $\sigma_i(\rho_y) = \sigma_y$ into account, one can find

$$\sigma_i(R) = \sigma_i(\rho_y) \frac{\rho_y^3}{R^3} = \sigma_y \frac{\rho_y^3}{R^3}. \tag{24}$$

The rate of corrosion at the outer surface ($\rho = R$) is defined by Eq. (2). Substituting Eq. (24) into Eq. (2) gives for $t \geq t^e$

$$\frac{dR}{dt} = -\left(a_R + m_R \, \sigma_y \, \frac{\rho_y^3}{R^3} \right). \tag{25}$$

The radius ρ_y is to be determined from Eq. (19) with (23) for r:

$$\ln \rho_y^3 - \frac{\rho_y^3}{R^3} = \frac{3\Delta p}{2\sigma_y} - 1 + 3 \ln \left[r^e + v_y \,(t - t^e) \right], \tag{26}$$

where r^e, v_y are defined by Eqs. (14), (22), respectively. For any $t > t^e$, the radius ρ_y can be found by a trial-and-error procedure.

Thus, for lifetime assessment of the ideal elastic-plastic sphere under pressure it is necessary to solve the simultaneous equations (25), (26) with "initial" condition (15). Lifetime is then defined as the time t^f at which $\rho_y = R$ and unhindered plastic flow begins. Using (8) and (23), complete yielding criterion (21) can be written in the form

$$R(t) = \exp\left(\frac{\Delta p}{2\sigma_y} \right) \left[r^e + v_y \,(t - t^e) \right].$$

The final time $t = t^f$ satisfying the required equation is the lifetime of the vessel in the sense involved.

In the general case, the problem is to be solved numerically. For some situations, closed-form solutions of the problem have been derived. For a power hardening elastic-plastic thick-walled spherical shell we can combine this theory with the solution presented in [13].

The dependence of the shell's lifetime on the initial data has been analyzed. It has been observed that, if the initial stress is close to the yield stress then the time of plastic-zone propagation throughout the vessel's wall can be much greater than the length of the pure elastic stage. Thus, the plastic-zone progress should be taken into account for lifetime assessment.

5 CONCLUSIONS

- Analytical solution for a pressurized thick-walled spherical shell of an ideal elastic-plastic material under general mechanochemical corrosion conditions has been found.

- Formulas have been obtained for assessment of the time of the initial yielding at the bore of the sphere and the time of fully plastic yielding.

- For the case of homogeneous stress, vessel's lifetime has been determined as the time to complete dissolution.

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